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An universal relation between fractal and Euclidean (topological) dimensions of random systems

A. Bershadskii[a]

P.O. Box 39953, Ramat-Aviv 61398, Tel-Aviv, Israel

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Abstract. It is shown that a dimension-invariant form $D(d) = bd^{\gamma}$ for fractal dimension D of random systems (where d is Euclidean dimension of the embedding space) is in good agreement with results of numerical simulations performed by different authors for critical ($p = p_c$) and subcritical ($p < p_c$) percolation, for lattice animals, and for different aggregation processes.

PACS. 64.60.Ak Renormalization-group, fractal, and percolation studies of phase transitions – 61.43.Hv Fractals; macroscopic aggregates (including diffusion-limited aggregates) – 05.40.+j Fluctuation phenomena, random processes, and Brownian motion

Explicit expressions for exponents as a function of dimensionalities are being discussed since a quarter of a century (Stanley-Betts formula), without an exact result. In particular, the problem of existence of an universal relation between fractal dimension of random systems (D) and Euclidean (or topological) dimension of embedding space (d) is still open to debates. To approximate results of numerical simulations of diffusion-limited aggregation, for instance, some different types of the dependence of D on d were suggested (see [1] for a review). When one is dealing with a specific type of data (as it is at the above example [1]) it may be difficult to chose from different approximation giving more or less good fit of the data. Therefore we should seek a dependence which gives good approximation of the data obtained for different types of processes (if such universal dependence exists, of course). In this note we shall represent a dependence $D(d)$ which may turn out to be appropriate for this role (in some interval of values of d). This dependence has a general nature (related to dimension-invariance) and gives good fit of the data obtained in numerical simulations of critical ($p = p_c$) and subcritical ($p < p_c$) percolation [2], lattice animals [3], cluster-cluster aggregation (with and without random impact) [4], and kinetic aggregation [5]. The dimension-invariant form of this dependence $D \simeq bd^{\gamma}$ (where b and γ are some constants) allows a simple comparison with the numerical data (Fig. 1). Under condition $D(1) = 1$ the constant $b = 1$. It is interesting that the exponent γ for the above mentioned systems belongs to rather narrow interval $0.6 < \gamma < 0.7$ (and only for diffusion-limited aggregation $\gamma \simeq 1$ [6]). It should be also noted that the dimension-invariant representation with $\gamma > 0$ is no longer valid above the upper critical dimension (if it exists) where the fractal dimensions become independent of d.

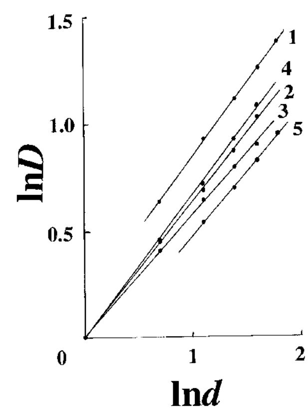

Fig. 1. $\ln D$ against $\ln d$ for data obtained at numerical simulations of critical ($p = p_c$) percolation: (1) [2], subcritical percolation and lattice animals; (2) [2,3], two types of cluster-cluster aggregation: with (3), and without (4), random impact [4], and of kinetic aggregation (5) [5]. Straight lines are drawn for comparison with dimension-invariant representation (13).

Let a set of balls with different sizes l cover the embedding space. Let us chose a subset with "homogeneous" mass distribution, *i.e.* for this subset the balls mass $M(l) \propto l^d$. The sizes of the balls in this subset have some

[a] e-mail: bersh@hotmail.com

statistical distribution. If the system under consideration is homogeneous in whole, then the average mass of the balls from this subset

$$\langle M(l)\rangle \propto \langle l^d\rangle \propto \langle l\rangle^d. \qquad (1)$$

For fractal systems

$$\langle M(l)\rangle \propto \langle l^d\rangle \propto \langle l\rangle^D \qquad (2)$$

where generally fractal dimension $D < d$.

If we are dealing with a dimension-invariant system [7,8], then we can find a general form of a function $g(q)$ in the multifractal relationship

$$\langle l^q\rangle \propto \langle l\rangle^{g(q)} \qquad (3)$$

where q is an arbitrary number.

Indeed, let us introduce dimensionless moments [7,8]

$$F_{qm} = \frac{\langle l^q\rangle}{\langle l^m\rangle^{q/m}} \qquad (4)$$

and generalized scaling

$$F_{qm} \sim F_{pm}^{\rho(q,p,m)}. \qquad (5)$$

Using (3) we obtain for the exponent $\rho(q,p,m)$ representation

$$\rho(q,p,m) = \frac{g(q) - g(m)q/m}{g(p) - g(m)p/m}\cdot \qquad (6)$$

Dimension-invariance of the system implies [7,8]

$$\rho(\alpha q, \alpha p, \alpha m) = \rho(q,p,m) \qquad (7)$$

(where α is an arbitrary positive number), or in terms of (6)

$$\frac{g(\alpha q) - g(\alpha m)q/m}{g(\alpha p) - g(\alpha m)p/m} = \frac{g(q) - g(m)q/m}{g(p) - g(m)p/m}\cdot \qquad (8)$$

It is easy to show that general solution of equation (8) is

$$g(q) = aq + bq^{\gamma} \qquad (9)$$

or

$$g(q) = aq + bq\ln q \qquad (10)$$

where a, b, and γ are some constants.

Comparing equations (2, 3) and (9, 10) we obtain general dimension-invariant forms of the dependence $D(d)$

$$D(d) = ad + bd^{\gamma} \qquad (11)$$

or

$$D(d) = ad + bd\ln d. \qquad (12)$$

Let us consider a particular solution (*cf.* (11)).

$$D(d) = bd^{\gamma}. \qquad (13)$$

In Figure 1 we show data of numerical simulations performed for critical ($p = p_c$) and sub-critical ($p < p_c$) percolation [2], lattice animals [3], for two types (with and without random impact) of cluster-cluster aggregation with linear trajectories [4] and for kinetic aggregation [5]. Log-log scales are chosen in this figure for comparison with dimension-invariant representation (13) (straight lines). It is interesting that for all systems represented in Figure 1 the exponents γ belong to a rather narrow interval $0.6 < \gamma < 0.7$. To find these values of the exponent γ from first principles seems to be an interesting problem for future investigations.

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